MICROCOPY RESOLUTION TEST CHART

ON CONSECUTIVE-k-OUT-OF-n: F SYSTEMS

BY

FUNG-YEE CHAN, LAI K. CHAN, and GWO DONG LIN

TECHNICAL REPORT NO. 389

MAY 11, 1987

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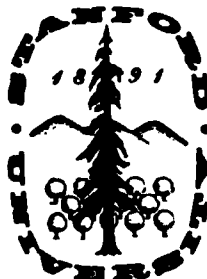JUN 15 1987
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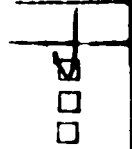
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A consecutive-k-out-of-n: F system consists of n linearly ordered components and the system fails if and only if at least k consecutive components fail. This type of system has attracted considerable attention since 1980. In this paper first a brief survey of the literature is given. Then the system is studied in the general case when the n components are statistically independent, but may not have the same reliability. Most of the papers in the literature use traditional combinatorial or conditional probability approaches to derive the formulae for the system reliability. Here we show that using the elegant and concise algebraic approach and network diagram representation given in Chapters 1 and 2 of Barlow and Proschan (1975), one can easily obtain the exact formula for the system reliability. Furthermore, we are able to obtain algorithms for finding all the minimal path sets and cut sets of the system, and hence the upper and lower bounds of the system reliability. A simplified upper bound is also proposed.

Kontoleon (1980) seems to be one of the first who studied the consecutive-k-out-of-n: F systems and described a computer algorithm for obtaining the system reliability. Then Chiang and Niu (1981) gave two practical examples, one about a telecommunication system and one about an oil pipeline system. Assuming that the component failure times are independently and identically distributed (i.i.d.), they presented a recursive formula to compute the exact reliability of the system. Bollinger and Salvia (1982) provided another recursive formula and also gave a practical example about the design of integrated circuit. Some direct computations or closed forms of the system reliability can be found in Bollinger (1982, 1984), Lambiris & Papastavridis (1985) and Chen & Hwang (1985).

Chao and Lin (1984) applied the concept of taboo probability to find a general closed-form formula for the system reliability. They studied the large system and proved that for $1 \leq k \leq 4$ the system reliability tends to $\exp\{-\lambda^k\}$ as $n \to \infty$ if all components have the same failure probability $\lambda n^{-1/k}$, where λ is a positive constant. Their conjecture that the above result also holds for the general case $k > 4$ is proved by Fu (1985).

Most of the works mentioned above are concerned with the case when all the components have the same reliability. For more general cases when the reliabilities are not identical, Derman et al. (1982) gave the upper and lower bounds of the system reliability. Shanthikumar (1982)

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and Hwang (1982) provided recursive formulas for the system reliability. Shanthikumar (1985) then extended his formula to a system whose components have exchangeable lifetimes.

The optimal sequencing problem is also of interest, i.e., how to arrange the components so that the system reliability is maximized. Let component #1 be the least reliable, component #2 the next least reliable and so on. Derman et al. (1982) conjectured that the optimal sequence for the case $k = 2$ is

$$(\#1, \#n, \#3, \#(n-2), \cdots, \#(n-3), \#4, \#(n-1), \#2)$$

which interlaces the more reliable components with the less reliable items. Wei et al. (1983) gave some partial solutions to this problem, and the complete solution is due to Malon (1984). As we can see in Malon (1985), the solution for the general case $k > 2$ is still unknown.

Bollinger and Salvia (1985) considered the dynamic case and developed a time-to-failure model when each component is exponentially distributed. Chen and Hwang (1985) also studied the system failure distribution when the component failure distributions are i.i.d. but may not be exponential.

The sections of this paper are arranged as follows. Section 1 gives a network diagram and a structure function representations of the consecutive-k-out-of-n: F systems. We have found that such representations, which were so nicely and concisely introduced in Chapters 1 and 2 of Barlow and Proschan (1975), are useful tools for the analyses of the system. Furthermore, results in their book can then be directly applied to the system. Perhaps the materials in their book should be more frequently used in future studies of the system. Section 2 derives a recursive formula for the system structure function, which is then used to obtain a recursive formula of the system reliabi lty by simply taking the expectation of the structure function. In Section 3, necessary and sufficient conditions for the n components forming a minimal path vector are given. Based on these conditions, a simple algorithm for finding all the minimal path sets is obtained. Upper and lower bounds of the system reliability are given in Section 4. A numerical example is also given in Section 4.

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1. Structure Function.

The notations used here are similar to those in Chapters 1 and 2 in Barlow and Proschan (1975). Using these elegant and concise algebraic and network diagram representations, we can gain inside into the characteristics of the system structure, and then obtain the system reliability and other properties.

The consecutive-k-out-of-n: F system can be represented by the following series – parallel network in which components with the same number i are identical. Such a representation is useful for visualizing some characteristics of the system (e.g., each column is a minimal cut set, and a path set can be obtained by selecting at least one functioning component from each column) and is helpful in understanding the algebraic proofs of the theorems.

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Put Figure 1 here
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Let the random variable

$$z_i = \begin{cases} 1 & \text{if component } i \text{ functions} \\ 0 & \text{if component } i \text{ fails} \end{cases}$$

and the structure function

$$\phi_n(z_n) = \begin{cases} 1 & \text{if the system functions} \\ 0 & \text{if the system fails} , \end{cases}$$

where the vector $z_n = (x_1, \cdots, x_n)$. Then from Figure 1 we can easily see that each column of parallel components $\{j, j+1, \cdots, j+k-1\} \equiv K_j$ is a minimal cut set. The structure function of the system can easily be seen to be

$$\phi_n(z_n) = \prod_{j=1}^{n-k+1} \coprod_{i \in K_j} x_i, \tag{1.1}$$

where $\coprod_{i \in K_j} x_i \equiv 1 - (1 - x_j)(1 - x_{j+1}) \cdots (1 - x_{j+k-1})$. The minimal path sets will be studied in Section 3. The following section gives a recursive formula for the structure function. Hereafter, except for special mention we adopt Boolean algebra when operating the numerical-value addition and multiplication.

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2. A Recursive Formula for the Exact Reliability.

From the structure function (1.1) we can derive the following interesting recursive formulas for the structure function (Theorem 1) and for the system reliability (Corollary 1). Note that in Theorem 1 we do not assume the statistical independence of n components and that $z_{n-k}, \cdots, z_n$ are seperated from z_{n-k+1} in the last term.

Theorem 1. $\phi_n(z_n) = \phi_{n-1}(z_{n-1}) - \phi_{n-k-1}(z_{n-k-1})\{z_{n-k}(1 - z_{n-k+1}) \cdots (1 - z_n)\}$.

Proof. By Figure 1 we see that

$$
\begin{aligned}
\phi_n(z_n) &= \phi_{n-1}(z_{n-1})\{1 - (1 - z_{n-k+1})(1 - z_{n-k+2}) \cdots (1 - z_n)\} \\
&= \phi_{n-1}(z_{n-1})\{1 - z_{n-k}(1 - z_{n-k+1}) \cdots (1 - z_n) \\
&\quad - (1 - z_{n-k})(1 - z_{n-k+1}) \cdots (1 - z_n)\},
\end{aligned}
\tag{2.1}
$$

where ϕ_{n-1} is the structure function of the system consisting of the first $n-k$ columns of Figure 1, and $z_{n-1} = (z_1, z_2, \cdots, z_{n-1})$. If $\phi_{n-1}(z_{n-1}) = 1$, then at least one of $z_{n-k}, z_{n-k+1}, \cdots, z_{n-1}$ is 1 which implies that

$$
\phi_{n-1}(z_{n-1})\{((1 - z_{n-k})(1 - z_{n-k+1}) \cdots (1 - z_n)\} \equiv 0.
$$

Therefore from (2.1)

$$
\phi_n(z_n) = \phi_{n-1}(z_{n-1})\{1 - z_{n-k}(1 - z_{n-k+1}) \cdots (1 - z_n)\}.
\tag{2.2}
$$

We can express

$$
\begin{aligned}
\phi_{n-1}(z_{n-1}) &= \left(\prod_{j=1}^{n-2k} \coprod_{i \in K_j} z_i \right) \left(\prod_{j=n-2k+1}^{n-k} \coprod_{i \in K_j} z_i \right) \\
&= \phi_{n-k-1}(z_{n-k-1}) \left(\prod_{j=n-2k+1}^{n-k} \coprod_{i \in K_j} z_i \right).
\end{aligned}
$$

Therefore (2.2) becomes

$$
\begin{aligned}
\phi_n(z_n) &= \phi_{n-1}(z_{n-1}) \\
&\quad - \phi_{n-k-1}(z_{n-k-1})\{z_{n-k}(1 - z_{n-k+1}) \cdots (1 - z_n)\} \left(\prod_{j=n-2k+1}^{n-k} \coprod_{i \in K_j} z_i \right)
\end{aligned}
\tag{2.3}
$$

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It remains to show that

$$z_{n-k}(1 - z_{n-k+1})\cdots(1 - z_n) = z_{n-k}(1 - z_{n-k+1})\cdots(1 - z_n) \prod_{j=n-2k+1}^{n-k} \coprod_{i \in K_j} z_i. \qquad (2.4)$$

If the LHS of (2.4) is equal to 1, then $z_{n-k} = 1$ and it can be easily seen from Figure 1 that $\prod_{j=n-2k+1}^{n-k} \coprod_{i \in K_j} z_i = 1$. Therefore (2.4) holds whatever LHS is 0 or 1. The proof is complete.

Assume that the n components are statistically independent, and let $p_i = P(z_i = 1)$ be the reliability of component i, $i = 1, 2, \cdots, n$, and $q_i = 1 - p_i$. Then the system reliability is

$$h_n(\boldsymbol{p}_n) = E\phi_n(\boldsymbol{z}_n) = P(\phi_n(\boldsymbol{z}_n) = 1),$$

where $\boldsymbol{p}_n = (p_1, p_2, \cdots, p_n)$. It is clear, for example, that $h_k(\boldsymbol{p}_k) = 1 - \prod_{i=1}^{k} q_i$, where $\boldsymbol{p}_k = (p_1, p_2, \cdots, p_k)$. Taking the expectation on both sides of Theorem 1, we obtain the following recursive formula for system reliability $h_n(\boldsymbol{p}_n)$ through the $h_j(\boldsymbol{p}_j)$, where $h_j(\boldsymbol{p}_j)$ is the system reliability of the consecutive-k-out-of-j: F system consisting of the first j components of the original system. We set $h_j \equiv 1$ if $j < k$.

Corollary 1. $h_n(\boldsymbol{p}_n) = h_{n-1}(\boldsymbol{p}_{n-1}) - h_{n-k-1}(\boldsymbol{p}_{n-k-1})p_{n-k} \prod_{i=n-k+1}^{n} q_i.$

The above formula was also obtained by Hwang (1982) using a different approach.

3. Minimal Path Sets.

A path vector is a vector $\boldsymbol{z}_n$ such that $\phi_n(\boldsymbol{z}_n) = 1$. It is called a minimal path vector if $\boldsymbol{z}_n^\circ < \boldsymbol{z}_n$ implies that $\phi_n(\boldsymbol{z}_n^\circ) = 0$, where $\boldsymbol{z}_n^\circ < \boldsymbol{z}_n$ means that $z_i^\circ \leq z_i$ for $i = 1, 2, \cdots, n$ with at least one inequality being strict. The set of components corresponding to those z_i with value 1 of a minimal path vector is called a minimal path set. Then from the definition of the system we have the following necessary and sufficient conditions for a minimal path vector.

Theorem 2. For $2 \leq k \leq n$, $\boldsymbol{z}_n$ is a minimal path vector of the consecutive-k-out-of-n: F system if and only if

$$z_i + z_{i+1} + \cdots + z_{i+k-1} \geq 1, \qquad i = 1, 2, \cdots n - k + 1 \qquad (3.1)$$

and

$$z_{i-1} \left(\sum_{i \leq j_1 < j_2 \leq i+k-1} z_{j_1} z_{j_2} \right) = 0, \qquad i = 1, 2, \cdots, n, \qquad (3.2)$$

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where $x_0 \equiv 1$, $x_{n+1} \equiv 1$, and $x_{n+i} \equiv 0$ for $i = 2, \cdots, k - 1$.

Proof. Equation (3.1) is the necessary and sufficient condition that x_n is a path vector. Equation (3.2) is the condition that there are no redundant functioning components.

So one way to search for *all* the minimal path vectors is to find all the solutions to (3.1) and (3.2). A more tractable method of finding all the minimal path vectors is given in Theorems 3 and 4. Given a system of n components, let the polynomial (a Boolean function) z_i represent that only component i functions while the remaining $n - 1$ components fail, and let the polynomial $z_i z_j$ represent that only the components i and j function. And the polynomial $z_i + z_j$ means that it consists of two polynomials z_i and z_j. The multiplications and additions of three or more polynomials are similarly defined. So, for example, considering a consecutive-2-out-of-3: F system, the polynomial $z_1 z_2$ means that components 1 and 2 function but component 3 fails (i.e., $x_1 = x_2 = 1$ and $x_3 = 0$) and $z_1 z_2 + z_3$ consists of $z_1 z_2$ and z_3.

Given a system with n components, let the polynomial $\Phi_n(x_n)$ represent the class of all the minimal path sets of the system, where $x_n = (z_1, \cdots, z_n)$. Consider the consecutive-k-out-of-k: F system with $k \geq 2$. Each polynomial $z_j (j = 1, 2, \cdots, n)$ represents a minimal path set. In other words, x_k is a minimal path if and only if the number of functioning components is one. So the polynomial $\Phi_k(x_k) = \sum_{j=1}^{k} z_j$ consists of all the minimal path sets. For $n \leq 2k$, it is easy to find $\Phi_n(x_n)$ as shown in the following theorem.

Theorem 3. For $k \geq 2$ and $n = k + m$ with $1 \leq m \leq k$,

$$\Phi_n(x_n) = \sum_{i=1}^{m} \sum_{j=1}^{i} z_i z_{k+j} + \sum_{i=m+1}^{k} z_i,$$

where the second summation is zero if $m = k$.

Proof. If x_n is a minimal path vector, then one and only one of its components $x_1, x_2, \cdots, x_k$ is 1. For $1 \leq i \leq m$, if $x_i = 1$ then one and only one of $x_{k+1}, x_{k+2}, \cdots, x_{k+i}$ is 1 and the other $n - 2$ components of x_n are 0. All such minimal path vectors can be represented by $\sum_{i=1}^{m} \sum_{j=1}^{i} z_i z_{k+j}$. For $m + 1 \leq i \leq k$ (if $m < k$), if $x_i = 1$ then other $n - 1$ components are 0. All such minimal path vectors can be represented by $\sum_{i=m+1}^{k} z_i$. The theorem is then proved.

For systems with $n \geq 2k \geq 4$, the following Theorem 4 gives an algorithm to generate all the minimal path sets of a system of $n + 1$ componetns from a system of n components with

the same k. The component $n - k + 1$ plays a major role in this algorithm. Let the polynomial $f_j(z_1, \cdots, z_n)$ represent a minimal path set of the consecutive-k-out-of-n: F system. Define the mapping Φ^* on the class of all f_j by

$$\Phi^*(f_j) = f_j, \quad \text{if } f_j \text{ does not contain } z_{n-k+1} \tag{3.3a}$$

$$= f_j \sum_{i=\ell+k+1}^{n+1} z_i, \quad \text{if } f_j \text{ contains } z_{n-k+1} \text{ and } z_\ell \text{ for some } \ell \text{ in}$$

$$\{n - 2k + 1, \cdots, n - k\}. \tag{3.3b}$$

The following figures would be helpful in understanding the proof of the theorem. In Figure 2a, the first $n - k + 1$ columns form a consecutive-k-out-of-n: F system and all the $n - k + 2$ columns form a consecutive-k-out-of-$n + 1$: F system. In Figure 2b, the first system consists of the first n components and the second system consists of all the $n + 1$ components.

$$\boxed{\text{Put Figures 2a and 2b here.}}$$

Since f_j represents a minimal path set, in (3.3a) it contains exactly one of $z_{n-k+2}, \cdots, z_n$ (see Fig. 2a), while in (3.3b) it does not contain any one of these polynomials (see Fig. 2a) but it contains exactly one z_ℓ, where $\ell \in \{n - 2k + 1, \cdots, n - k\}$ (see Fig. 2b).

Theorem 4. Let $n \geq 2k \geq 4$, then the class of all minimal path sets of a consecutive-k-out-of-$n + 1$: F system can be represented by

$$\Phi_{n+1}(z_1, \cdots, z_{n+1}) = \Phi^*(f_1) + \Phi^*(f_2) + \cdots + \Phi^*(f_p),$$

where $f_1, f_2, \cdots, f_p$ represent all the minimal path sets of the consecutive-k-out-of-n: F sytem which consists of the first n components.

Proof. Let $A_{n,s}$ (resp. $A_{n+1,s}$) be the class of minimal path vectors of the consecutive-k-out-of-n (resp. $n + 1$): F system with $x_{n-k+1} = s$, $s = 0, 1$.

First we proceed to show that (3.3a) is a mapping from $A_{n,0}$ onto $A_{n+1,0}$. Let $\boldsymbol{x}_n \in A_{n,0}$. Then $x_{n-k+i} = 1$ for a unique $i \in \{2, \cdots, k\}$ (see Fig. 2a), i.e., f_j contains exactly one of $z_{n-k+2}, \cdots, z_n$. If we let $\boldsymbol{x}_{n+1}^*$ be such that $x_i^* = x_i$, $i = 1, 2, \cdots, n$ and $x_{n+1}^* = 0$. Then

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$x_{n+1}^{\bullet}$ satisfies (3.1) of Theorem 2 and (3.2) for $n + 1$ components if we set $x_{n+1}^{\bullet} \equiv 1$ and $x_{n+3}^{\bullet} \equiv \cdots \equiv x_{n+k}^{\bullet} \equiv 0$. So $x_{n+1}^{\bullet} \in A_{n+1,0}$. Therefore, (3.3a) is a mapping from $A_{n,0}$ into $A_{n+1,0}$. On the other hand, for any $x_{n+1}^{\bullet} \in A_{n+1,0}$, $x_{n-k+i}^{\bullet} = 1$ for a unique $i \in \{2, \cdots, k\}$ and $x_{n+1}^{\bullet} = 0$. If we let x_n be such that $x_i = x_i^{\bullet}$ for $i = 1, 2, \cdots, n$, then x_n satisfies (3.1) and (3.2) of Theorem 2 (see Fig. 2a) and hence it belongs to $A_{n,0}$. Furthermore, one can obtain $x_{n+1}^{\bullet}$ from x_n through the mapping (3.3a).

Now we proceed to show that (3.3b) is a mapping from $A_{n,1}$ onto $A_{n+1,1}$. For any $x_n \in A_{n,1}$, $x_{n-k+1} = 1$ and hence $x_{n-k+i} = 0$ for $i = 2, \cdots, k$ (see Fig. 2a). Let ℓ be the last functioning component before component $n - k + 1$. Then we have $n - 2k + 1 \leq \ell \leq n - k$ and component ℓ is the only functioning component among the components $n - 2k + 1, \cdots, n - k$ (see Fig. 2b). For each $t = \ell + k + 1, \cdots, n + 1$, let $x_{n+1}^{\bullet}(t) = (x_1^{\bullet}, \cdots, x_{n+1}^{\bullet})$ be such that $x_m^{\bullet} = x_m$ if $m = 1, 2, \cdots, n - k + 1$, $x_t^{\bullet} = 1$ and $x_m^{\bullet} = 0$ if $m \notin \{1, 2, \cdots, n - k + 1, t\}$. Then $x_{n+1}^{\bullet}(t)$ satisfies (3.1) and (3.2) of Theorem 2 for a system of $n + 1$ components (see Fig. 2b) and hence $x_{n+1}^{\bullet}(t) \in A_{n+1,1}$. So $x_{n+1}^{\bullet}(t)$, $t = \ell + k + 1, \cdots, n + 1$, are the elements in $A_{n+1,1}$ generated by $x_n \in A_{n,1}$, and such a mapping can be represented by (3.3b). Now it remains to be shown that for each $x_{n+1}^{\bullet} \in A_{n+1,1}$, there exists $x_n \in A_{n,1}$ such that $x_{n+1}^{\bullet}$ can be obtained from x_n through the mapping (3.3b). First, note that $x_\ell^{\bullet} = 1$ for a unique $\ell \in \{n - 2k + 1, \cdots, n - k\}$ and that $x_t^{\bullet} = 1$ for a unique $t \in \{\ell + k + 1, \cdots, n + 1\}$ (see Fig. 2b). So if we let $x_m = x_m^{\bullet}$ for $m = 1, 2, \cdots, n - k + 1$ and $x_m = 0$ for $m = n - k + 2, \cdots, n$, then $x_n = (x_1, \cdots, x_n)$ satisfies (3.1) and (3.2) of Theorem 2 and hence $x_n \in A_{n,1}$ (see Fig. 2b). Furthermore, we note that $x_{n+1}^{\bullet}$ can be obtained from x_n through the mapping represented by (3.3b).

Therefore all minimal path vectors of a system of $n + 1$ components can be obtained from all those of a system of n components through (3.3a) and (3.3b). The theorem is then proved.

Theorem 5. For each minimal path set f of the consecutive-k-out-of-n: F system, the number of functioning components, denoted by order (f), satisfies

$$[\frac{n}{k}] \leq \text{ order } (f) \leq \begin{cases} 2[\frac{n+1}{k+1}] - 1, & \text{if } [\frac{n+1}{k+1}] = \frac{n+1}{k+1} \\ 2[\frac{n+1}{k+1}], & \text{if } [\frac{n+1}{k+1}] < \frac{n+1}{k+1}, \end{cases}$$

where $[a]$ denotes the largest ingeter $\leq a$

Proof. Partition the system into consecutive blocks of k components, starting from component

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1 (see Fig. 3a). The last block contains less than k components if $[n/k] < n/k$. For any minimal path set f, each block contains at least one of its functioning components, with the exception of the last block which may not contain any if $[n/k] < n/k$. Hence order $(f) \geq [n/k]$, which is achieved by the minimal path set with only components $k, 2k, \cdots, [n/k]k$ functioning. To find the upper bound of order (f), we try to put as many functioning components as possible in the system. Consider the case $[(n+1)/(k+1)] = (n+1)/(k+1)$. The minimal path set in Fig. 3b has $2[(n+1)/(k+1)] - 1$ functioning components. If we partition the system into blocks with the first block having k components and each of the remaining blocks having $k+1$ components, then for any minimal path set, the first block has exactly one functioning component, and each of the remaining blocks cannot have more than two components. Hence the order of any minimal path set is less than or equal to that of the minimal set given in Fig. 3b. The case $[(n+1)/(k+1)] < (n+1)/(k+1)$ can be similarly proved except to note that the last block cannot have more than one functioning component for any minimal path set.

Put Figures 3a, 3b and 3c here

4. Bounds of the System Reliability.

Although we can find the exact reliability of the consecutive-k-out-of-n: F system, e.g., by Corollary 1, the recursive computations often lose accuracy. So we study in this section the lower and upper bounds of the system reliability, which can be computed directly.

Assume that the components are statistically independent and $p_i = P(x_i = 1)$, $q_i = 1 - p_i$, $i = 1, 2, \cdots, n$. Then from Figure 1 we have all the $n - k + 1$ minimal cut sets and hence the lower bound of system reliability $h_n(p_n)$,

$$\prod_{j=1}^{n-k+1} \left(1 - \prod_{i=j}^{j+k-1} q_i \right) \leq h_n(p_n) \tag{4.1}$$

(Barlow and Proschan (1975), p. 35).

On the other hand, if we find all the minimal path sets of the system by Theorems 3 or

4, say $\{P_j\}_{j=1}^p$, then we can obtain an upper bound of system reliability as follows.

$$h_n(\underline{p})_n \leq 1 - \prod_{j=1}^p \left(1 - \prod_{i \in P_j} p_i\right). \tag{4.2}$$

Here we propose another simplified upper bound of the system reliability. Let E_j be the event that the system fails and the number of failed components is exactly j. Then, assuming $n \geq k + 1$,

$$1 - h_n(p_n) = P\left(\bigcup_{j=k}^n E_j\right) = \sum_{j=k}^n P(E_j) \geq P(E_k) + P(E_{k+1}).$$

Therefore,

$$h_n(p_n) \leq 1 - [P(E_k) + P(E_{k+1})]. \tag{4.3}$$

Now we proceed to calculate the probabilities $P(E_k)$ and $P(E_{k+1})$. It is clear from Figure 1 that E_k is the union of $n - k + 1$ disjoint sub-events, say $E_{k,r}$, $r = 1, 2, \cdots, n - k + 1$, where $E_{k,r}$ means that the components $r, r + 1, \cdots, r + k - 1$ fail and the rest function. So we have

$$P(E_k) = \sum_{r=1}^{n-k+1} P(E_{k,r}) = \sum_{r=1}^{n-k+1} \left[\prod_{j=r}^{r+k-1} q_j \prod_{t \notin \{r,\cdots,r+k-1\}} p_t\right].$$

Similarly, E_{k+1} is the union of $(n - k + 1)(n - k - 1) + 1$ sub-events, say $E_{k+1,r,s}$. Here $r = 1, 2, \cdots, n - k + 1$; $s = 1, 2, \cdots, n - k - 1$ for $r = 1, 2, \cdots, n - k$, and $s = 1, 2, \cdots, n - k$ for $r = n - k + 1$. Also $E_{k+1,r,s}$ means that the components $r, r + 1, \cdots, r + k - 1$ fail, component $r + k$ funtions ($x_{n+1} \equiv 1$ if necessary), component $s, s \notin \{r, \cdots, r + k\}$, fails and all the remaining components function. So we have

$$P(E_{k+1}) = \sum_{r=1}^{n-k+1} \left\{ \left(\prod_{j=r}^{r+k-1} q_j\right) p_{r+k} \left[\sum_{s \notin \{r,\cdots,r+k\}} \left(q_s \prod_{\substack{t \notin \{s\} \cup \\ \{r,\cdots,r+k\}}} p_t\right)\right]\right\},$$

in which $p_{n+1} \equiv 1$.

The upper bound (4.3) can be sharpened by including $P(E_{k+2}), P(E_{k+3}), \cdots$. But if the p_i values are large, the gain would be too small to compensate the complexity in the calculation of $P(E_{k+2}), \cdots$. This is because one may reason that it is quite unlikely that

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too many components will fail at the same time. As an example, consider a consecutive-2-out-of-6: F system, $h_6(p_6) = 1 - P(E_2) - \cdots - P(E_6)$. When all $p_i = 0.9$, $h_6 = 0.954261$ while $p(E_4) = 1.215 \times 10^{-3}$, $p(E_5) = 5.4 \times 10^{-5}$ and $p(E_6) = 10^{-6}$. When all $p_i = 0.99$, $h_6 = 0.999504029601$ while $p(E_4) = 1.47015 \times 10^{-7}$, $p(E_5) = 5.94 \times 10^{-10}$ and $p(E_6) = 10^{-12}$.

However, when p is small, $1 - P(E_k) - P(E_{k+1})$ is close to 1 and hence is not a good upper bound. On the other hand, $P(E_{n-1}) + P(E_n)$ is large and hence has significant effect on the upper bound. Therefore a better upper bound of $h_n(p_n)$ for both small and large p values could be achieved by including

$$P(E_{n-1}) = \sum_{i=1}^{n} p_i \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} q_j \right) \quad \text{and} \quad P(E_n) = \prod_{j=1}^{n} q_j.$$

So we have the following upper bound

$$h_n(p_n) \leq 1 - P(E_k) - P(E_{k+1}) - P(E_{n-1}) - P(E_n). \tag{4.4}$$

For the lower and upper bounds in the i.i.d. case, see Chiang and Niu (1981), Salvia (1982) and Fu (1985). We shall compare the upper bounds (4.2) and (4.4) to the following (4.5) proposed by Derman et al. (1982)

$$h_n(p_n) \leq 1 - \frac{\{E(N)\}^2}{E(N^2)}, \tag{4.5}$$

where N is the number of failed minimal cut sets,

$$E(N) = \sum_{i=1}^{n-k+1} \prod_{j=i}^{i+k-1} q_j$$

and

$$E(N^2) = E(N) + \sum_{\substack{i,j=1 \\ i \neq j}}^{n-k+1} \prod_{\substack{t \in \{i,\cdots,i+k-1\}\cup \\ \{j,\cdots,j+k-1\}}} q_t.$$

Example. Consider a consecutive-2-out-of-6: F system with statistically independent components (the special case when all $p_i = p$ and $q_i = q = 1 - p$ is also included).

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(a) By Corollary 1, the system reliability

$$h_6(\pmb{p}_6) = (1 - q_1 q_2)(1 - p_3 q_4 q_5 - p_4 q_5 q_6) - p_1 q_2 q_3 - p_2 q_3 q_4 + p_1 p_4 q_2 q_3 q_5 q_6$$

$$= 1 - 5q^2 + 4q^3 + 3p^4 - 4q^5 + q^6.$$

(b) By Theorem 5, the orders of the minimal path sets are between $[6/2] = 3$ and $2[7/3] = 4$. The orders 3 and 4 are achieved by the minimal path sets $z_2 z_4 z_6$ and $z_1 z_3 z_4 z_6$, respectively. By (4.1), a lower bound for $h_6(\pmb{p}_6)$ is

$$L_1 \equiv \prod_{j=1}^{6} (1 - q_j q_{j+1}) = (1 - q^2)^6.$$

(c) By (4.4), one upper bound for $h_6(\pmb{p}_6)$ is

$$U_1 \equiv 1 - [P(E_2) + P(E_3) + P(E_5) + P(E_6)],$$

where

$$P(E_2) = \sum_{r=1}^{6} \left[q_r q_{r+1} \prod_{t \neq r, r+1} p_t \right] = 5p^4 q^2,$$

$$P(E_3) = \sum_{r=1}^{6} \left\{ q_r q_{r+1} p_{r+2} \left[\sum_{s \notin \{r, r+1, r+2\}} \left(q_s \prod_{t \notin \{s, r, r+1, r+2\}} p_t \right) \right] \right\}$$

$$= 16p^3 q^3,$$

$$P(E_5) = \sum_{i=1}^{6} p_i \left(\prod_{\substack{j=1 \\ j \neq i}}^{6} q_j \right) = 6pq^5$$

and

$$P(E_6) = \prod_{j=1}^{6} q_j = q^6.$$

(d) By (4.5), another upper bound is

$$U_2 \equiv 1 - \{E(N)\}^2 / E(N^2),$$

where

$$E(N) = \sum_{i=1}^{6} q_i q_{i+1} = 5q^2$$

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and

$$E(N^2) = \sum_{i=1}^{5} q_i q_{i+1} + \sum_{i \neq j} \prod_{t \in \{i,i+1,j,j+1\}} q_t$$
$$= 5q^2 + 8q^3 + 12q^4.$$

(e) To find all the minimal path sets, first consider the consecutive-k-out-of-$2k$: F systems which consists of the first $2k = 4$ components of the original system. By Theorem 3, they are

$$f_1 = z_1 z_3, \quad f_2 = z_2 z_3, \quad f_3 = z_2 z_4.$$

By Theorem 4, all the minimal path sets of the consecutive-k-out-of-$(2k + 1)$: F system can be found through (z_3 plays a major role)

$$\Phi^*(f_1) = f_1(z_4 + z_5) = z_1 z_3 z_4 + z_1 z_3 z_5$$

$$\Phi^*(f_2) = f_2(z_5) = z_2 z_3 z_5$$

$$\Phi^*(f_3) = z_2 z_4.$$

Continuing the application of Theorem 4 to the consecutive-k-out-of-$(2k + 2)$: F system (with z_4 playing a major role), we obtain all the required minimal path sets:

$$z_1 z_3 z_4 z_6, \quad z_1 z_3 z_5, \quad z_2 z_3 z_5, \quad z_2 z_4 z_5, \quad z_2 z_4 z_6.$$

(f) By (4.2), an upper bound for $h_6(p_6)$ is

$$U_3 \equiv (1 - p_1 p_3 p_4 p_6)(1 - p_1 p_3 p_5)(1 - p_2 p_3 p_5)(1 - p_2 p_4 p_6)$$
$$= 1 - (1 - p^4)(1 - p^3)^4.$$

Figures 4a and 4b gives the curves for the values of h_6, L_1, U_1, U_2 and U_3 when $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = p$ for $0 \leq p \leq 1$. It is interesting to find that for a large component reliability, e.g., $p \geq 0.9$ (see Figs. 4a, 4b and Table 1 below), the simple upper bound U_1 is the best among U_1, U_2 and U_3. In fact, even the upper bound in (4.3) is better than both U_2 and U_3 for $p \geq 0.9$. For small component reliability, e.g., $p \leq 0.2$, the upper bound U_3 using minimal path sts is significantly better than U_1 and U_2 (see Fig. 4a and Table 1 below).

Put Figures 4a and 4b here.

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Table 1

Comparison of the Exact Reliability h_n with its Upper Bounds
U_1, U_2, U_3 and Lower Bound L_1 when the Component Reliability
p is Large or Small for a Consecutive-2-out-of-6: F System

p	Reliability				
	U_1	U_2	U_3	h_n	L_1
0.99	0.999504	0.999508	1.000000	0.999504	0.999500
0.98	0.998035	0.998064	0.999999	0.998032	0.998001
0.96	0.992299	0.992508	0.999973	0.992263	0.992025
0.94	0.983072	0.983705	0.999819	0.982900	0.982129
0.92	0.970678	0.972012	0.999320	0.970158	0.968407
0.90	0.955476	0.957770	0.998145	0.954261	0.950990
0.20	0.273984	0.161426	$3.316754\ e-2$	$2.822378\ e-2$	$6.046619\ e-3$
0.15	0.188524	0.117611	$1.393121\ e-2$	$1.234683\ e-2$	$1.645564\ e-3$
0.10	0.102196	0.076186	$4.093587\ e-3$	$3.781140\ e-3$	$2.476103\ e-4$
0.05	0.031031	0.037025	$5.061626\ e-4$	$4.869699\ e-4$	$8.810969\ e-6$
0.03	0.012058	0.021966	$1.088381\ e-4$	$1.063943\ e-4$	$7.210018\ e-7$
0.01	0.001445	0.007240	$4.053116\ e-6$	$4.053116\ e-6$	$3.120815\ e-9$

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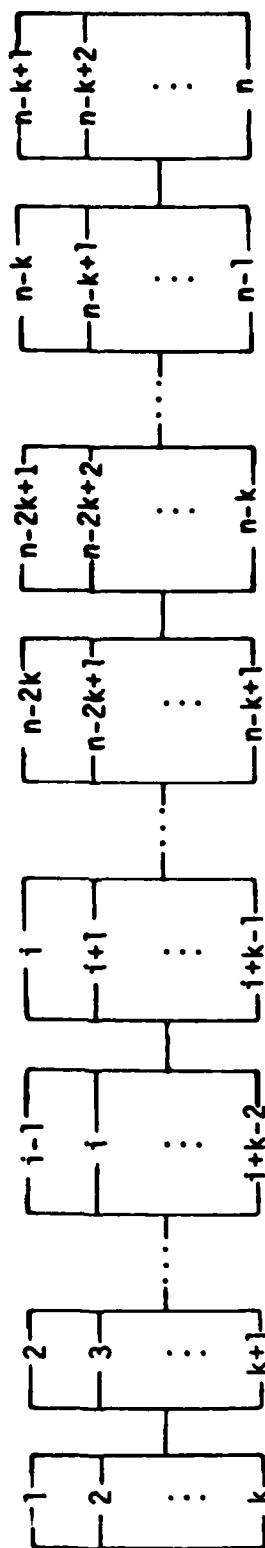
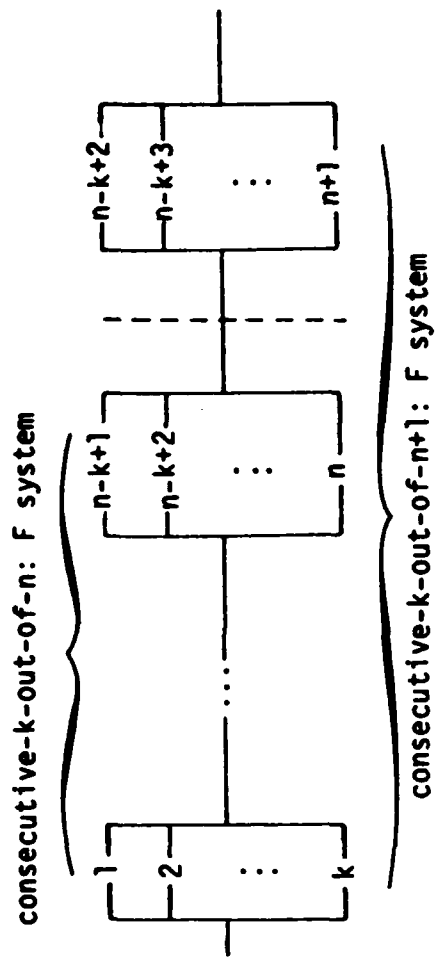FIGURE. 1. Network representation of the consecutive-k-out-of-n: F system.

FIGURE 2a

The system fails if k consecutive components fail

FIGURE 2b

$$1 \text{---}\cdots\text{---} k \text{---} k+1 \text{---}\cdots\text{---} 2k \text{---}\cdots\text{---} \cdots \text{---} \cdots \text{---} \cdots \text{---} [\tfrac{n}{k}]k \text{---} \cdots \text{---} n$$

x_1: $\underbrace{0 \cdots 0 \; 1}_{k}$ $\underbrace{0 \cdots 0 \; 1}_{k}$ $\cdots$ $\underbrace{0 \cdots 0 \; 1}_{k}$ $\underbrace{0 \cdots 0 \; 1}_{k}$ $\underbrace{0 \cdot \cdots 0}_{< k}$

FIGURE 3a

$$1 \text{---}\cdots\text{---} k \text{---} k+1 \text{---} k+2 \text{---}\cdots\text{---} 2k+1 \text{---} 2k+2 \text{---} 2k+3 \text{---}\cdots\text{---} 3k+2 \text{---}\cdots\text{---} \cdots \text{---} \cdots \text{---} n = [\tfrac{n+1}{k+1}](k+1)-1$$

x_1: $\underbrace{1 \; 0 \cdots 0}_{k}$ $\underbrace{1 \quad 1 \; 0 \cdots 0}_{k+1}$ $\underbrace{1 \quad 1 \; 0 \cdots 0}_{k+1}$ $\cdots$ $\underbrace{1 \quad 1 \; 0 \cdots 0}_{k+1}$

FIGURE 3b

$$1 \text{---}\cdots\text{---} k \text{---} k+1 \text{---} k+2 \text{---}\cdots\text{---} 2k+1 \text{---}\cdots\text{---} \cdots \text{---} \cdots \text{---} [\tfrac{n+1}{k+1}](k+1) \text{---} \cdots \text{---} n$$

x_1: $\underbrace{1 \; 0 \cdots 0}_{k}$ $\underbrace{1 \quad 1 \; 0 \cdots 0}_{k+1}$ $\cdots$ $\underbrace{1 \quad 1 \; 0 \cdots 0}_{k+1}$ $\underbrace{1 \; 0 \cdots 0}_{(n+1)-[\tfrac{n+k}{k+1}](k+1) \le k}$
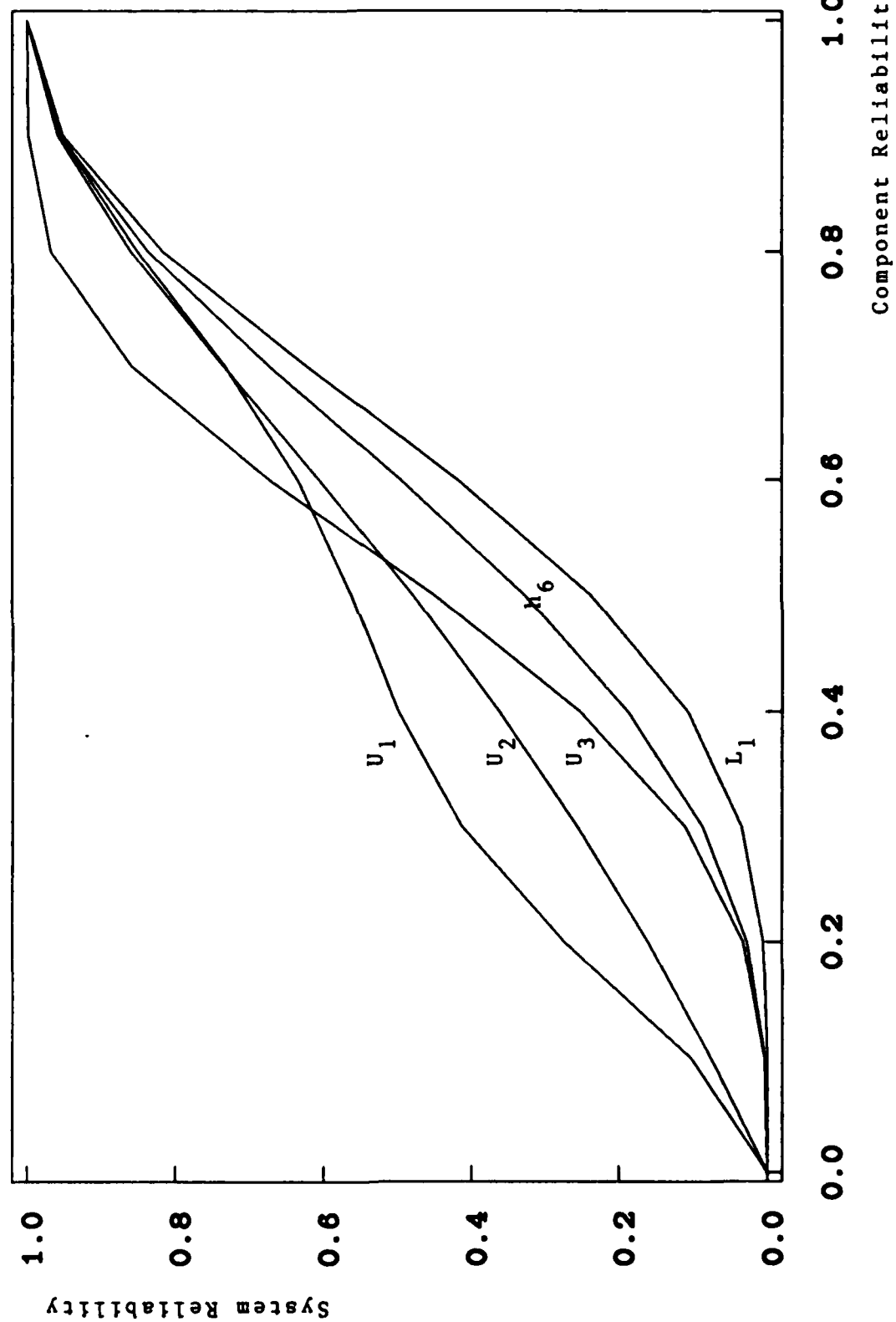
FIGURE 3c

FIGURE 4a

Comparison of the Exact System Reliability h_6 with its
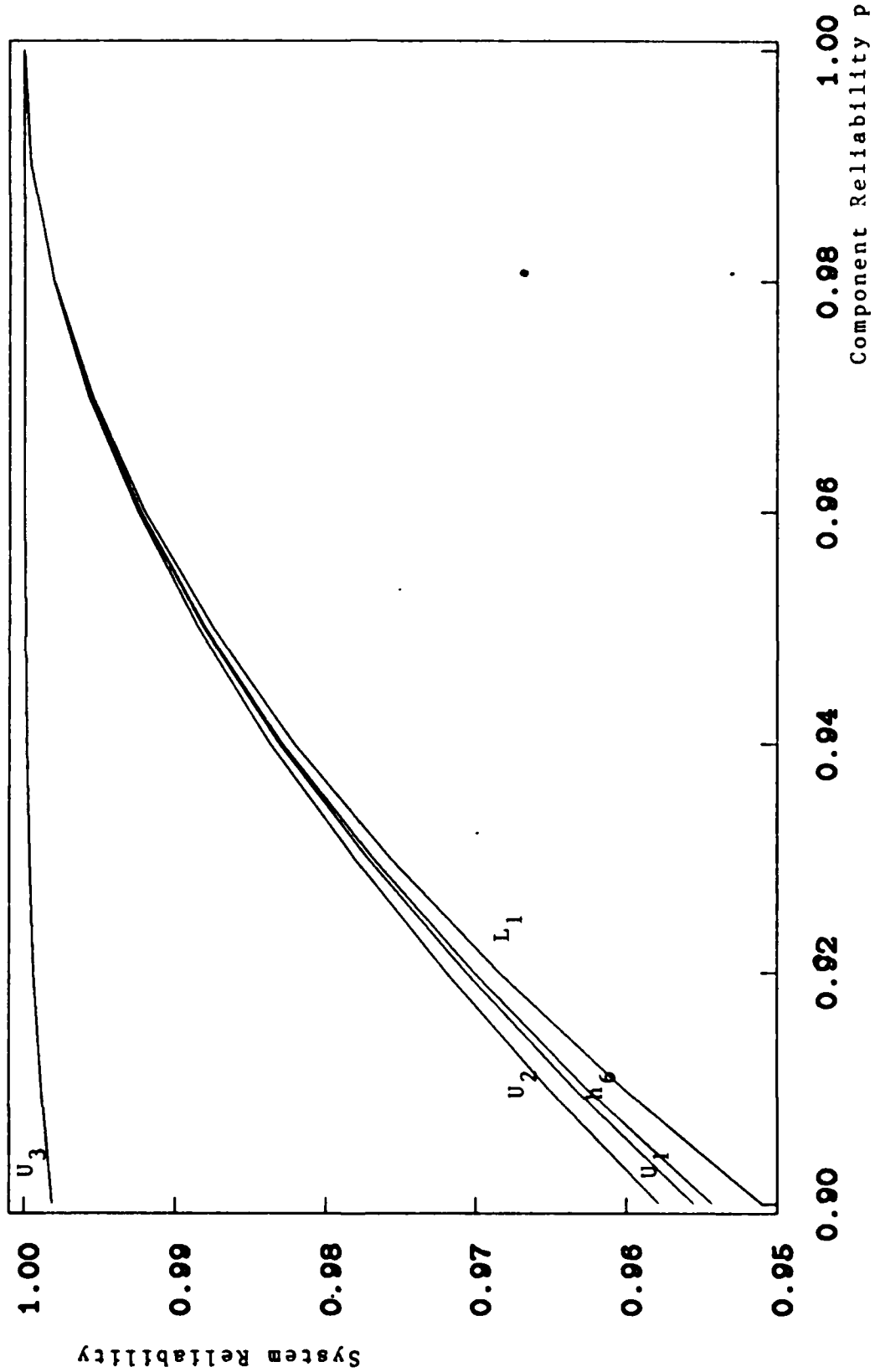Upper Bounds U_1, U_2, U_3 & Lower Bound L_1

FIGURE 4b

Comparison of the Exact System Reliability h_6 with its Upper Bounds U_1, U_2, U_3 & Lower Bound L_1 when p is Large

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1. REPORT NUMBER 389	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE *(and Subtitle)* On Consecutive-k-out-of-n: F Systems		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL REPORT
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Fung-Yee Chan, Lai K. Chan, and Gwo Dong Lin		8. CONTRACT OR GRANT NUMBER(s) N00014-86-K-0156
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Stanford University Stanford, CA 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR-042-267
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Statistics & Probability Program Code 1111		12. REPORT DATE May 11, 1987
		13. NUMBER OF PAGES 24
14. MONITORING AGENCY NAME & ADDRESS*(If different from Controlling Office)*		15. SECURITY CLASS. *(of this report)* UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT *(of this Report)* APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED		
17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)* System Reliability, Minimal Paths, Networks		

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

First a brief survey of the consecutive-k-out-of-n: F system is given. Through the use of structure function and using network diagrams to represent the system, the system reliability and algorithms for generating all the minimal path and cut sets are obtained. A lower bound and three upper bounds of the systems reliability are given.

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